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Note

The Numerical Solution of a Class of Abel Integral Equations by Piecewise Polynomials

THE METHOD OF PIECEWISE POLYNOMIALS

In the study of the thermodynamic states of axially symmetric radiating plasma columns, the connection between radiance and emission coefficient is given by an Abel integral equation of the form

$$I(x) = 2 \cdot \int_x^R \frac{f(r)\, r\, dr}{(r^2 - x^2)^{1/2}}, \qquad 0 \leqslant x \leqslant R. \tag{1}$$

Here, R denotes the radius of the plasma column. The radiance function $I(x)$ (with $I(x) = 0$, $x \geqslant R$) is a function of the lateral coordinate x and is measured spectroscopically by a line probe taken at a certain number of points along the x-axis (see, for example, [1, 4, 8, 10, 16]). Numerical methods for determining the unknown radial distribution of the emission coefficient $f(r)$ have been considered by a number of authors. Almost all of these methods are based on the numerical evaluation of the inversion formula for the given Eq. (1),

$$f(r) = -\frac{1}{\pi} \cdot \int_r^R \frac{I'(x)\, dx}{(x^2 - r^2)^{1/2}}, \qquad 0 \leqslant r \leqslant R. \tag{2}$$

(See Refs. [12, 18, 10, 16, 2, 1, 6, 9, 8, 7, 4, 11, 14, 15]. A comprehensive survey of such methods (up to 1966) is contained in [4]). More recently, some authors have proposed direct methods for solving (1). Linz [14] uses finite-difference techniques to approximate $f(r)$ at given uniformly spaced points; whereas, Weiss and Anderssen [20] and Weiss [19] apply so-called product integration which yields approximate values for $f(r)$ at discrete (not necessarily uniformly spaced) points. Their approach, however, does not allow the order of the method to go beyond a certain limit (see [19]). In this note we describe a direct method for solving the Abel integral Eq. (1) which makes use of spline functions where the usual continutity requirements are somewhat relaxed. To be precise, we shall only require continuty of the approximating function itself but not of any of its derivatives. To avoid confusion we call these functions in the following piecewise polynomials.

Let $m \geqslant 1$ be a given integer, and introduce the set of points Z by

$$Z = \{x = x_{k.0} : 0 = x_{N.0} < x_{N-1.0} < \cdots < x_{0.0} = R\},$$

with $N \geqslant 1$. We denote the set of all piecewise polynomials $s(r)$ of degree m and with knots Z by $S_m(Z)$. For $x_{k+1.0} \leqslant r \leqslant x_{k.0}$ $(k = 0, 1,..., N-1)$ $s(r) \in S_m(Z)$ shall have the representation

$$s(r) = s_k(r) = s_k(x_{k.0}) + \sum_{\nu=1}^{m} \frac{c_{k.\nu}}{\nu!} (r - x_{k.0})^\nu, \tag{3}$$

with

$$s_0(x_{0.0}) = s_0(R) = f(R) = 0.$$

Since $s(r)$ is to be continuous for all $0 \leqslant r \leqslant R$ we have

$$s_k(x_{k.0}) = s_{k-1}(x_{k.0}) = s_{k-1}(x_{k-1.m}), \qquad k = 1,..., N-1 \tag{4}$$

with $x_{k-1.m}$ defined below.

The unknown coefficients $\{c_{k.1},..., c_{k.m}\}$ in (3) will be computed recursively in the following manner. For a given value of k, define the points $\{x_{k.1},..., x_{k.m}\}$ by

$$x_{k.0} > x_{k.1} > \cdots > x_{k.m} = x_{k+1.0}\,.$$

We now require that $s_k(r)$, together with the known representations $s_{k-1}(r),..., s_0(r)$, satisfy the given integral equation (1) at these points:

$$2 \cdot \int_{x_{k.j}}^{R} \frac{s(r)\, r\, dr}{(r^2 - x_{k.j}^2)^{1/2}} = I(x_{k.j}), \qquad j = 1,..., m.$$

This relation may be rewritten as

$$2 \int_{x_{k.j}}^{x_{k.0}} \frac{s_k(r)\, r\, dr}{(r^2 - x_{k.j}^2)^{1/2}} = I(x_{k.j}) - 2 \cdot \sum_{\mu=0}^{k-1} \int_{x_{\mu+1.0}}^{x_{\mu.0}} \frac{s_\mu(r)\, r\, dr}{(r^2 - x_{k.j}^2)^{1/2}},$$

$$j = 1,..., m \; (k = 0,..., N-1). \tag{5}$$

For a given value of k relation (5) represents a system of m linear algebraic equations for the set of coefficients $\{c_{k.1},..., c_{k.m}\}$ of $s_k(r)$. The coefficient matrix of the system is given by the elements

$$I_{j.\nu}^{(k)} = \frac{2}{\nu!} \int_{x_{k.j}}^{x_{k.0}} \frac{r(r - x_{k.0})^\nu\, dr}{(r^2 - x_{k.j}^2)^{1/2}} \qquad (j, \nu = 1,..., m). \tag{6}$$

If integration by parts is carried out we obtain

$$I_{j.\nu}^{(k)} = -\frac{2}{(\nu-1)!}\int_{x_{k.j}}^{x_{k.0}} (r - x_{k.0})^{\nu-1} \cdot (r^2 - x_{k.j}^2)^{1/2}\, dr \quad (j, \nu = 1,..., m). \tag{7}$$

It follows by a standard result from the theory of interpolation (see, for example, [5, p. 26]) that for any choice of the points $\{x_{k.j}\}$ such that

$$x_{k+1.0} = x_{k.m} < x_{k.m-1} < \cdots < x_{k.1} < x_{k.0} \qquad (k = 0,..., N-1)$$

the matrices defined by (6) or (7) are all nonsingular.

We return now to the given Abel integral equation (1) and the underlying physical problem. Suppose that the values for the radiance function $I(x)$ have been obtained at the points $0 = z_M < z_{M-1} < \cdots < z_1 < z_0 = R$. Let $m \geqslant 1$ be given, and, for simplicity, assume that $M = N \cdot m$ for some positive integer N. The knots Z of $s(r) \in S_m(Z)$ are then given by setting

$$x_{k.0} = z_{km} \qquad (k = 0, 1,..., N).$$

For a given value of k the points $\{x_{k.j}\}$ are chosen as

$$z_{km} > z_{km+1} > \cdots > z_{km+m} = z_{(k+1)m} \qquad (k = 0, 1,..., N-1).$$

We observe that the degree m of the piecewise polynomials $s(r)$ need not be kept fixed over the given interval $0 \leqslant r \leqslant R$ but may be altered whenever the physical situation justifies this.

Results dealing with the convergence and the order of the method of piecewise polynomials will be given elsewhere (see also [3]). Here, we shall illustrate the application and the efficiency of the method by presenting a numerical example.

Numerical Results

For reasons of comparison we choose for the radiance function $I(x)$ in (1) the same function used by Cremers and Birkebak [4, p. 1059]. It is given at 31 uniformly spaced points in the interval $0 \leqslant x \leqslant 1$, as indicated in Table I. Piecewise polynomials of degree $m = 1$ (with the given points taken as the knots Z) are used to compute numerical values for the emission coefficient $f(r)$. The results are compared with those obtained by the method of Cremers and Birkebak [4] who used fourth-degree polynomials and least-squares techniques to compute $f(r)$ from formula (2).

In Table I we list the errors $e(x_{k.0})$ for the method of piecewise polynomials of degree one (column (I)) and for the method from [4] mentioned above (column (II)) at the knots $x_{k.0} = k/30$ $(k = 30, 29,..., 1, 0)$.

TABLE I

k	$e(x_{k,0})$ (I)	$e(x_{k,0})$ (II)	k	$e(x_{k,0})$ (I)	$e(x_{k,0})$ (I)I
30	0.0000	0.0000	14	0.0003	−0.0003
29	−0.0010	−0.0004	13	0.0004	−0.0004
28	−0.0008	0.0003	12	0.0006	−0.0003
27	−0.0005	0.0003	11	0.0007	−0.0001
26	−0.0008	−0.0001	10	0.0007	0.0029
25	−0.0005	−0.0004	9	0.0007	0.0013
24	−0.0005	−0.0004	8	0.0009	−0.0029
23	−0.0005	−0.0001	7	0.0017	−0.0073
22	−0.0002	0.0000	6	0.0015	−0.0035
21	−0.0004	−0.0002	5	0.0007	0.0036
20	0.0000	−0.0003	4	0.0002	0.0000
19	0.0000	−0.0004	3	−0.0001	−0.0012
18	−0.0000	−0.0003	2	−0.0003	−0.0012
17	−0.0001	0.0000	1	−0.0018	0.0003
16	0.0003	0.0000	0	−0.0029	0.0018
15	0.0003	−0.0002			

All the computations were performed on the CDC 6400 (single precision) at Dalhousie University Computer Center.

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